

Lagrangian and Hamiltonian structure for Constant Astigmatism Equation

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Abstract

In this paper we found a Lagrangian representation and corresponding Hamiltonian structure for the constant astigmatism equation. Utilizing this Hamiltonian structure and extra conservation law densities we construct first evolution commuting flow of the third order. Also, we present two simple solutions of the constant astigmatism equation expressed via elementary functions.

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1 Introduction

Plenty of integrable equations were found in classical differential geometry. One of them is the famous Bonnet equation also known as the sin-Gordon

equation. This equation expresses angles between asymptotic directions of surfaces of negative constant Gaussian curvature. Recently, interest to the Bonnet equation was renewed due to a successful search of integrable cases of Weingarten surfaces. The equation describing surfaces of constant astigmatism

$$u_{tt} + \left(\frac{1}{u}\right)_{xx} + 2 = 0 \quad (1)$$

was considered in a set of papers (see detail in [1]). Also, a transformation between the Bonnet equation and (1) was found. However this transformation is very sophisticated. It is not so easy to recompute solutions and Hamiltonian structures from the Bonnet equation to (1). By this reason, we construct the Lagrangian, corresponding Hamiltonian structure, first evolution commuting flow of the third order and two simple solutions of (1) in this paper. The inverse transformation (from (1) back to the Bonnet equation) is not so complicated. We believe that our results can be effectively utilized in the theory of the Bonnet equation.

2 Lagrangian and Hamiltonian Structure

The Lagrangian

$$S = \int \left(\frac{1}{2} \Omega_{xx} \Omega_{tt} - f(\Omega, \Omega_x, \Omega_{xx}, \dots) \right) dx dt \quad (2)$$

determines the Euler–Lagrange equation

$$\Omega_{xxtt} = \frac{\delta \mathbf{F}}{\delta \Omega}, \quad (3)$$

where $\mathbf{F} = \int f(\Omega, \Omega_x, \Omega_{xx}, \dots) dx$. Obviously, two local conservation laws (of the energy and of the momentum) can be obtained (due to E. Noether’s Theorem) from the energy-momentum tensor. For instance, the conservation law of the momentum is

$$(\Omega_{xx} \Omega_{xt})_t = \left(\frac{1}{2} \Omega_{xt}^2 + \Omega_x \Omega_{xtt} - G \right)_x,$$

where $G_x = \frac{\delta \mathbf{F}}{\delta \Omega} \Omega_x$, while the conservation law of the energy is

$$\left(\frac{1}{2} \Omega_{xt}^2 + f(\Omega, \Omega_x, \Omega_{xx}, \dots) \right)_t = (\Omega_t \Omega_{xtt} + Q)_x,$$

where $Q_x = -\frac{\delta F}{\delta \Omega} \Omega_t + \frac{\partial f}{\partial \Omega} \Omega_t + \frac{\partial f}{\partial \Omega_x} \Omega_{xt} + \frac{\partial f}{\partial \Omega_{xx}} \Omega_{xxt} + \dots$

If, for instance, $f(\Omega, \Omega_x, \Omega_{xx})$, then

$$(\Omega_{xx}\Omega_{xt})_t = \left(\frac{1}{2}\Omega_{xt}^2 + \Omega_x\Omega_{xtt} - f + \frac{\partial f}{\partial \Omega_x}\Omega_x - \left(\frac{\partial f}{\partial \Omega_{xx}} \right)_x \Omega_x + \frac{\partial f}{\partial \Omega_{xx}}\Omega_{xx} \right)_x,$$

$$\left(\frac{1}{2}\Omega_{xt}^2 + f(\Omega, \Omega_x, \Omega_{xx}) \right)_t = \left(\Omega_{xtt}\Omega_t + \frac{\partial f}{\partial \Omega_x}\Omega_t - \left(\frac{\partial f}{\partial \Omega_{xx}} \right)_x \Omega_t + \frac{\partial f}{\partial \Omega_{xx}}\Omega_{xt} \right)_x.$$

Euler–Lagrange equation (3) can be written as a Hamiltonian system

$$\Omega_t = \partial_x^{-1} \frac{\delta \mathbf{H}}{\delta w}, \quad w_t = \partial_x^{-1} \frac{\delta \mathbf{H}}{\delta \Omega}, \quad \mathbf{H} = \int (f(\Omega, \Omega_x, \Omega_{xx}) + \frac{1}{2}w^2)dx,$$

where $w = \Omega_{xt}$.

In this paper we consider just the case $f(\Omega, \Omega_x, \Omega_{xx}) = -2\Omega - \ln \Omega_{xx}$. Corresponding Euler–Lagrange equation (3) is nothing but constant astigmatism equation (1), where $u = \Omega_{xx}$. Thus, constant astigmatism equation (1) possesses the local Lagrangian representation

$$S = \int \left(\frac{1}{2}\Omega_{xx}\Omega_{tt} + \ln \Omega_{xx} + 2\Omega \right) dxdt, \quad (4)$$

two local conservation laws (the momentum and the energy, respectively):

$$(\Omega_{xx}\Omega_{xt})_t = \left(\frac{1}{2}\Omega_{xt}^2 + \Omega_x\Omega_{xtt} + 2\Omega - \frac{\Omega_x\Omega_{xxx}}{\Omega_{xx}^2} + \ln \Omega_{xx} \right)_x,$$

$$\left(\frac{1}{2}\Omega_{xt}^2 - 2\Omega - \ln \Omega_{xx} \right)_t = \left(\Omega_{xtt}\Omega_t - \frac{\Omega_{xxx}}{\Omega_{xx}^2}\Omega_t - \frac{\Omega_{xt}}{\Omega_{xx}} \right)_x$$

and non-local Hamiltonian structure

$$\Omega_t = \partial_x^{-1} \frac{\delta \mathbf{H}}{\delta w}, \quad w_t = \partial_x^{-1} \frac{\delta \mathbf{H}}{\delta \Omega}, \quad (5)$$

where the Hamiltonian $\mathbf{H} = \int (\frac{1}{2}w^2 - 2\Omega - \ln \Omega_{xx})dx$ and the momentum $\mathbf{P} = \int \Omega_{xx}w dx$.

Also constant astigmatism equation (1) has extra two conservation laws

$$\partial_t \sqrt{4u + \left(\frac{u_x}{u} \pm u_t \right)^2} = \pm \partial_x \sqrt{\frac{4}{u} + \left(\frac{u_x}{u^2} \pm \frac{u_t}{u} \right)^2}.$$

Thus, one can construct a third order symmetry

$$u_y = \partial_x \frac{\delta \tilde{\mathbf{H}}}{\delta w}, \quad w_y = \partial_x \frac{\delta \tilde{\mathbf{H}}}{\delta u}, \quad \tilde{\mathbf{H}} = \int \sqrt{4u + \left(\frac{u_x}{u} \pm w_x \right)^2} dx,$$

which can be derived from the same Hamiltonian structure (1) (let us remind that $u = \Omega_{xx}$) but with another Hamiltonian density $\tilde{h} = \sqrt{\left(\frac{\Omega_{xxx}}{\Omega_{xx}} \pm w_x\right)^2 + 4\Omega_{xx}}$. Then this third order symmetry (let us remind that $w = \Omega_{xt}$) reduces to two three dimensional equations

$$\Omega_{xxt} + 2\Omega_y \sqrt{\frac{\Omega_{xx}}{1 - \Omega_y^2}} \pm \frac{\Omega_{xxx}}{\Omega_{xx}} = 0, \quad \Omega_{yt} = \sqrt{\frac{1 - \Omega_y^2}{\Omega_{xx}}} \pm \frac{\Omega_{xy}}{\Omega_{xx}}. \quad (6)$$

The compatibility condition $(\Omega_{xxt})_y = (\Omega_{yt})_{xx}$ leads to the equation

$$\left(\frac{\Omega_{xxx}}{\Omega_{xx}} \pm 2\Omega_y \sqrt{\frac{\Omega_{xx}}{1 - \Omega_y^2}}\right)_y + \left(\frac{\Omega_{xy}}{\Omega_{xx}} \pm \sqrt{\frac{1 - \Omega_y^2}{\Omega_{xx}}}\right)_{xx} = 0,$$

which is again nothing but an Euler–Lagrange equation associated with the local Lagrangian representation (cf. (4))

$$\tilde{S} = \int [\Omega_{xy} \ln \Omega_{xx} \pm 2\sqrt{\Omega_{xx}(1 - \Omega_y^2)}] dx dy.$$

One can express Ω_y from the first equation in (6) and substitute it into the second equation in (6). This gives again the constant astigmatism equation.

3 Simple Solutions

It is well known that one of most important features of integrable systems is existence of infinitely many solutions associated with so called multidimensional theta functions. Shortly speaking, multi-phase (or multi-gap) solutions of such equations as the Bonnet equation cannot be expressed via elementary functions except their degenerations. Moreover, even a self-similar solution of the Bonnet equation is determined by a Painlevé equation, which cannot be expressed via elementary functions.

In this Section we present just two most simple (but nontrivial) solutions of (1), which are expressed via elementary functions. So, we would like to illustrate such an unusual quality of this integrable system here.

It is easy to see that a self-similar solution $u(x, t) = t^k f(xt^m)$ for constant astigmatism equation (1) exists just for $k = 2$ and $m = 1$. A corresponding ordinary differential equation is

$$\left(\frac{1}{f}\right)'' + z^2 f''(z) + 4zf'(z) + 2f(z) + 2 = 0.$$

Its general solution is

$$f(z) = -\frac{1}{2}z^{-2} \left(z^2 + Az + B \pm \sqrt{z^4 + 2Az^3 + (A^2 + 2B - 4)z^2 + 2ABz + B^2} \right).$$

Thus, constant astigmatism equation (1) possesses the solution

$$u = -\frac{1}{2}x^{-2} \left(x^2t^2 + Axt + B \pm \sqrt{x^4t^4 + 2Ax^3t^3 + (A^2 + 2B - 4)x^2t^2 + 2ABxt + B^2} \right).$$

Also, the travelling wave solution $u(\theta)$, where $\theta = kx - \omega t$, reduces constant astigmatism equation (1) to the ordinary differential equation

$$\omega^2 u'' + k^2 \left(\frac{1}{u} \right)'' + 2 = 0,$$

whose general solution is given by

$$u(x, t) = \frac{1}{2} \frac{k}{\omega} \left[\frac{k\omega}{4} c_0 - \frac{1}{k\omega} (kx - \omega t - \theta_0)^2 \right] \pm \frac{k}{\omega} \sqrt{\frac{1}{4} \left[\frac{k\omega}{4} c_0 - \frac{1}{k\omega} (kx - \omega t - \theta_0)^2 \right]^2 - 1}.$$

It will be interesting to recompute these both solutions into the Bonnet equation. We hope to consider this problem in forthcoming paper.

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